

The Jacobi inversion formula

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Abstract

We look for differential equations satisfied by the generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$\frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha + 1)\Gamma(\beta + 1)}(1 - x)^{\alpha}(1 + x)^{\beta} + M\delta(x + 1) + N\delta(x - 1),$$

where $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$.

In order to find explicit formulas for the coefficients of these differential equations we have to solve systems of equations of the form

$$\sum_{i=1}^{\infty} A_i(x) D^i P_n^{(\alpha,\beta)}(x) = F_n(x), \quad n = 1, 2, 3, \dots,$$

where the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are independent of n . This system of equations has a unique solution given by

$$A_i(x) = 2^i \sum_{j=1}^i \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{i+1}} P_{i-j}^{(-\alpha-i-1, -\beta-i-1)}(x) F_j(x), \quad i = 1, 2, 3, \dots$$

This is a consequence of the Jacobi inversion formula

$$\sum_{k=j}^i \frac{\alpha + \beta + 2k + 1}{(\alpha + \beta + k + j + 1)_{i-j+1}} \times \\ \times P_{i-k}^{(-\alpha-i-1, -\beta-i-1)}(x) P_{k-j}^{(\alpha+j, \beta+j)}(x) = \delta_{ij}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots,$$

which is proved in this paper.

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1 Introduction

In [14] T.H. Koornwinder introduced the generalized Jacobi polynomials $\left\{P_n^{\alpha,\beta,M,N}(x)\right\}_{n=0}^{\infty}$ which are orthogonal on the interval $[-1, 1]$ with respect to the weight function

$$\frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(\alpha + 1)\Gamma(\beta + 1)}(1 - x)^{\alpha}(1 + x)^{\beta} + M\delta(x + 1) + N\delta(x - 1),$$

where $\alpha > -1$, $\beta > -1$, $M \geq 0$ and $N \geq 0$. As a limit case he also found the generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$ which are orthogonal on the interval $[0, \infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha + 1)}x^{\alpha}e^{-x} + M\delta(x),$$

where $\alpha > -1$ and $M \geq 0$. These generalized Jacobi polynomials and generalized Laguerre polynomials are related by the limit

$$L_n^{\alpha,M}(x) = \lim_{\beta \rightarrow \infty} P_n^{\alpha,\beta,0,M}\left(1 - \frac{2x}{\beta}\right).$$

In [6] we proved that for $M > 0$ the generalized Laguerre polynomials satisfy a unique differential equation of the form

$$M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0,$$

where $\{a_i(x)\}_{i=0}^{\infty}$ are continuous functions on the real line and $\{a_i(x)\}_{i=1}^{\infty}$ are independent of the degree n . In [1] H. Bavinck found a new method to obtain the main result of [6]. This inversion method was found in a similar way as was done in [4] in the case of generalizations of the Charlier polynomials. See also section 4 for more details. In [8] we used this inversion method to find all differential equations of the form

$$\begin{aligned} M \sum_{i=0}^{\infty} a_i(x)y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x)y^{(i)}(x) + \\ + MN \sum_{i=0}^{\infty} c_i(x)y^{(i)}(x) + xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0, \end{aligned} \quad (1)$$

where the coefficients $\{a_i(x)\}_{i=1}^{\infty}$, $\{b_i(x)\}_{i=1}^{\infty}$ and $\{c_i(x)\}_{i=1}^{\infty}$ are independent of n and the coefficients $a_0(x)$, $b_0(x)$ and $c_0(x)$ are independent of x , satisfied by the Sobolev-type Laguerre polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^{\infty}$ which are orthogonal with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\Gamma(\alpha + 1)} \int_0^{\infty} x^{\alpha} e^{-x} f(x)g(x)dx + Mf(0)g(0) + Nf'(0)g'(0),$$

where $\alpha > -1$, $M \geq 0$ and $N \geq 0$. These Sobolev-type Laguerre polynomials $\left\{L_n^{\alpha,M,N}(x)\right\}_{n=0}^{\infty}$ are generalizations of the generalized Laguerre polynomials $\left\{L_n^{\alpha,M}(x)\right\}_{n=0}^{\infty}$. In fact we have

$$L_n^{\alpha,M,0}(x) = L_n^{\alpha,M}(x) \quad \text{and} \quad L_n^{\alpha,0}(x) = L_n^{(\alpha)}(x).$$

In this paper we will prove an inversion formula involving the classical Jacobi polynomials which can be used to find differential equations of the form

$$M \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + N \sum_{i=0}^{\infty} b_i(x) y^{(i)}(x) + MN \sum_{i=0}^{\infty} c_i(x) y^{(i)}(x) + (1-x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) = 0, \quad (2)$$

where the coefficients $\{a_i(x)\}_{i=1}^{\infty}$, $\{b_i(x)\}_{i=1}^{\infty}$ and $\{c_i(x)\}_{i=1}^{\infty}$ are independent of n and the coefficients $a_0(x)$, $b_0(x)$ and $c_0(x)$ are independent of x , satisfied by the generalized Jacobi polynomials $\{P_n^{\alpha, \beta, M, N}(x)\}_{n=0}^{\infty}$. In [7] we applied the special case $\beta = \alpha$ of this inversion formula to solve the systems of equations obtained in [11].

The inversion formula for the Charlier polynomials obtained in [4] (see also section 4) was also used in [2] to find difference operators with Sobolev-type Charlier polynomials as eigenfunctions.

In [3] H. Bavinck and H. van Haeringen used similar inversion formulas to find difference equations for generalized Meixner polynomials.

2 The classical Laguerre and Jacobi polynomials

In this section we list the definitions and some properties of the classical Laguerre and Jacobi polynomials which we will use in this paper. For details the reader is referred to [5], [13] and [17].

The classical Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ can be defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}, \quad n = 0, 1, 2, \dots \quad (3)$$

for all α . Their generating function is given by

$$(1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n \quad (4)$$

and for all $n \in \{0, 1, 2, \dots\}$ we have

$$D^i L_n^{(\alpha)}(x) = (-1)^i L_{n-i}^{(\alpha+i)}(x), \quad i = 0, 1, 2, \dots, n, \quad (5)$$

where $D = \frac{d}{dx}$ denotes the differentiation operator. The Laguerre polynomials satisfy the linear second order differential equation

$$xy''(x) + (\alpha + 1 - x)y'(x) + ny(x) = 0. \quad (6)$$

It is well-known that

$$\frac{x^n}{n!} = \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} L_k^{(\alpha)}(x), \quad n = 0, 1, 2, \dots \quad (7)$$

This formula can easily be proved by using definition (3) and changing the order of summation as follows

$$\begin{aligned}
\sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} L_k^{(\alpha)}(x) &= \sum_{k=0}^n (-1)^k \binom{n+\alpha}{n-k} \sum_{j=0}^k (-1)^j \binom{k+\alpha}{k-j} \frac{x^j}{j!} \\
&= \sum_{j=0}^n \sum_{k=0}^{n-j} (-1)^k \binom{n+\alpha}{n-j-k} \binom{j+k+\alpha}{k} \frac{x^j}{j!} \\
&= \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{x^j}{j!} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} = \frac{x^n}{n!}, \quad n = 0, 1, 2, \dots
\end{aligned}$$

The classical Jacobi polynomials $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$ can be defined by

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \frac{(n+\alpha+\beta+1)_k}{k!} \frac{(\alpha+k+1)_{n-k}}{(n-k)!} \left(\frac{x-1}{2}\right)^k, \quad n = 0, 1, 2, \dots \quad (8)$$

$$= (-1)^n \sum_{k=0}^n \frac{(-n-k-\alpha-\beta)_k}{k!} \frac{(-n-\alpha)_{n-k}}{(n-k)!} \left(\frac{x-1}{2}\right)^k, \quad n = 0, 1, 2, \dots \quad (9)$$

$$= 2^{-n} \sum_{k=0}^n \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (x-1)^k (x+1)^{n-k}, \quad n = 0, 1, 2, \dots \quad (10)$$

for all α and β . For all $n \in \{0, 1, 2, \dots\}$ we have

$$D^i P_n^{(\alpha, \beta)}(x) = \frac{(n+\alpha+\beta+1)_i}{2^i} P_{n-i}^{(\alpha+i, \beta+i)}(x), \quad i = 0, 1, 2, \dots, n. \quad (11)$$

These Jacobi polynomials satisfy the linear second order differential equation

$$(1-x^2)y''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]y'(x) + n(n + \alpha + \beta + 1)y(x) = 0. \quad (12)$$

Further we have for $\alpha + \beta + 1 > 0$ (compare with [15], page 277, formula (30))

$$\left(\frac{1-x}{2}\right)^n = \sum_{k=0}^n \frac{(-n)_k (\alpha + k + 1)_{n-k} (\alpha + \beta + 2k + 1)}{(\alpha + \beta + k + 1)_{n+1}} P_k^{(\alpha, \beta)}(x), \quad n = 0, 1, 2, \dots \quad (13)$$

This formula is much less known than formula (7) for the Laguerre polynomials. However, the proof is quite similar. In section 5 we will prove a much more general formula.

We remark that (13) can be written in a more general form as

$$\begin{aligned}
&\sum_{k=0}^n \frac{(-n)_k (\alpha + \beta + 1)_k (\alpha + k + 1)_{n-k} (\alpha + \beta + 2k + 1)}{\Gamma(\alpha + \beta + n + k + 2)} P_k^{(\alpha, \beta)}(x) \\
&= \frac{1}{\Gamma(\alpha + \beta + 1)} \left(\frac{1-x}{2}\right)^n, \quad n = 0, 1, 2, \dots,
\end{aligned} \quad (14)$$

which is valid for all α and β .

The Jacobi polynomials $\{P_n^{(\alpha, \beta)}(x)\}_{n=0}^{\infty}$ and the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ are related by the limit

$$L_n^{(\alpha)}(x) = \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)}\left(1 - \frac{2x}{\beta}\right). \quad (15)$$

We remark that if we replace x by $1 - \frac{2x}{\beta}$ in (13), multiply by β^n and let β tend to infinity in the complex plane along the halfline where $\alpha + \beta$ is real and $\alpha + \beta + 1 > 0$ we obtain (7) by using (15) and the fact that we have for all $n \in \{0, 1, 2, \dots\}$

$$(-n)_k(\alpha + k + 1)_{n-k} = (-1)^k \binom{n + \alpha}{n - k} n!, k = 0, 1, 2, \dots, n.$$

3 The systems of equations

Let $\alpha > -1$. The Sobolev-type Laguerre polynomials $\{L_n^{\alpha, M, N}(x)\}_{n=0}^{\infty}$ can be written as

$$L_n^{\alpha, M, N}(x) = A_0 L_n^{(\alpha)}(x) + A_1 D L_n^{(\alpha)}(x) + A_2 D^2 L_n^{(\alpha)}(x), \quad n = 0, 1, 2, \dots,$$

where the coefficients A_0 , A_1 and A_2 are given by

$$\begin{cases} A_0 = 1 + M \binom{n + \alpha}{n - 1} + \frac{n(\alpha + 2) - (\alpha + 1)}{(\alpha + 1)(\alpha + 3)} N \binom{n + \alpha}{n - 2} + \\ \quad + \frac{MN}{(\alpha + 1)(\alpha + 2)} \binom{n + \alpha}{n - 1} \binom{n + \alpha + 1}{n - 2} \\ A_1 = M \binom{n + \alpha}{n} + \frac{n - 1}{\alpha + 1} N \binom{n + \alpha}{n - 1} + \frac{2MN}{(\alpha + 1)^2} \binom{n + \alpha}{n} \binom{n + \alpha + 1}{n - 2} \\ A_2 = \frac{N}{\alpha + 1} \binom{n + \alpha}{n - 1} + \frac{MN}{(\alpha + 1)^2} \binom{n + \alpha}{n} \binom{n + \alpha + 1}{n - 1}. \end{cases}$$

For details concerning these Sobolev-type Laguerre polynomials and their definition the reader is referred to [9] and [12]. Since the classical Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfy the differential equation (6) it is quite reasonable to look for differential equations of the form (1) for these Sobolev-type Laguerre polynomials $\{L_n^{\alpha, M, N}(x)\}_{n=0}^{\infty}$ in view of this definition and the fact that $L_n^{\alpha, 0, 0}(x) = L_n^{(\alpha)}(x)$. In [8] it is shown that this leads to eight systems of equations for the coefficients $\{a_i(x)\}_{i=0}^{\infty}$, $\{b_i(x)\}_{i=0}^{\infty}$ and $\{c_i(x)\}_{i=0}^{\infty}$. In order to find these coefficients we have to solve systems of equations which are of the form

$$\sum_{i=1}^{\infty} A_i(x) D^{i+k} L_n^{(\alpha)}(x) = F_n(x), \quad n = k + 1, k + 2, k + 3, \dots,$$

where $k \in \{0, 1, 2, \dots\}$ and the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are independent of n . In [8] it is pointed out that this system of equations has a unique solution given by

$$A_i(x) = (-1)^{i+k} \sum_{j=1}^i L_{i-j}^{(-\alpha-i-k-1)}(-x) F_{j+k}(x), \quad i = 1, 2, 3, \dots$$

This is an easy consequence of the Laguerre inversion formula

$$\sum_{k=j}^i L_{i-k}^{(-\alpha-i-1)}(-x) L_{k-j}^{(\alpha+j)}(x) = \delta_{ij}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots, \quad (16)$$

which was found by H. Bavinck in [1]. For more details the reader is referred to [1] and [8]. See also section 4 of this paper.

Now we take $\alpha > -1$ and $\beta > -1$. The generalized Jacobi polynomials $\{P_n^{\alpha,\beta,M,N}(x)\}_{n=0}^{\infty}$ can be written as

$$P_n^{\alpha,\beta,M,N}(x) = A_0 P_n^{(\alpha,\beta)}(x) + [A_1(1-x) - A_2(1+x)] DP_n^{(\alpha,\beta)}(x), \quad n = 0, 1, 2, \dots,$$

where the coefficients A_0 , A_1 and A_2 are given by

$$\left\{ \begin{aligned} A_0 &= 1 + M \frac{\binom{n+\beta}{n-1} \binom{n+\alpha+\beta+1}{n}}{\binom{n+\alpha}{n}} + N \frac{\binom{n+\alpha}{n-1} \binom{n+\alpha+\beta+1}{n}}{\binom{n+\beta}{n}} + \\ &\quad + MN \frac{(\alpha+\beta+2)^2}{(\alpha+1)(\beta+1)} \binom{n+\alpha+\beta+1}{n-1}^2 \\ A_1 &= \frac{M}{\alpha+\beta+1} \frac{\binom{n+\beta}{n} \binom{n+\alpha+\beta}{n}}{\binom{n+\alpha}{n}} + \frac{MN}{\alpha+1} \binom{n+\alpha+\beta}{n-1} \binom{n+\alpha+\beta+1}{n} \\ A_2 &= \frac{N}{\alpha+\beta+1} \frac{\binom{n+\alpha}{n} \binom{n+\alpha+\beta}{n}}{\binom{n+\beta}{n}} + \frac{MN}{\beta+1} \binom{n+\alpha+\beta}{n-1} \binom{n+\alpha+\beta+1}{n}. \end{aligned} \right.$$

Here we used the same definition as in [14], but in a slightly different notation. The case $\alpha + \beta + 1 = 0$ must be understood by continuity. In view of this definition and the fact that the classical Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ satisfy the differential equation (12) it is quite natural to look for differential equations of the form (2) satisfied by these generalized Jacobi polynomials as was already pointed out in [10]. Again this leads to eight systems of equations for the coefficients $\{a_i(x)\}_{i=0}^{\infty}$, $\{b_i(x)\}_{i=0}^{\infty}$ and $\{c_i(x)\}_{i=0}^{\infty}$. In order to find these coefficients we have to solve systems of equations which are of the form

$$\sum_{i=1}^{\infty} A_i(x) D^i P_n^{(\alpha,\beta)}(x) = F_n(x), \quad n = 1, 2, 3, \dots,$$

where the coefficients $\{A_i(x)\}_{i=1}^{\infty}$ are independent of n . This system of equations has a unique solution given by

$$A_i(x) = 2^i \sum_{j=1}^i \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{i+1}} P_{i-j}^{(-\alpha-i-1, -\beta-i-1)}(x) F_j(x), \quad i = 1, 2, 3, \dots$$

This is a consequence of the Jacobi inversion formula

$$\sum_{k=j}^i \frac{\alpha + \beta + 2k + 1}{(\alpha + \beta + k + j + 1)_{i-j+1}} \times$$

$$\times P_{i-k}^{(-\alpha-i-1, -\beta-i-1)}(x) P_{k-j}^{(\alpha+j, \beta+j)}(x) = \delta_{ij}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots, \quad (17)$$

which will be proved in this paper. Again, the case $\alpha + \beta + 1 = 0$ must be understood by continuity. We remark that if we replace x by $1 - \frac{2x}{\beta}$ in (17), multiply by β^{i-j} and let β tend to infinity along the positive real axis we obtain the Laguerre inversion formula (16) by using (15).

In [11] we found all differential equations of the form

$$M \sum_{i=0}^{\infty} a_i(x) y^{(i)}(x) + (1 - x^2) y''(x) - 2(\alpha + 1) x y'(x) + n(n + 2\alpha + 1) y(x) = 0, \quad (18)$$

where $\{a_i(x)\}_{i=0}^{\infty}$ are continuous functions on the real line and $\{a_i(x)\}_{i=1}^{\infty}$ are independent of n , satisfied by the symmetric generalized ultraspherical polynomials $\left\{P_n^{\alpha, \alpha, M, M}(x)\right\}_{n=0}^{\infty}$ defined by

$$P_n^{\alpha, \alpha, M, M}(x) = C_0 P_n^{(\alpha, \alpha)}(x) - C_1 x D P_n^{(\alpha, \alpha)}(x), \quad n = 0, 1, 2, \dots,$$

where

$$\begin{cases} C_0 = 1 + \frac{2Mn}{\alpha + 1} \binom{n + 2\alpha + 1}{n} + 4M^2 \binom{n + 2\alpha + 1}{n - 1}^2 \\ C_1 = \frac{2M}{2\alpha + 1} \binom{n + 2\alpha}{n} + \frac{2M^2}{\alpha + 1} \binom{n + 2\alpha}{n - 1} \binom{n + 2\alpha + 1}{n} \end{cases}.$$

We remark that these polynomials form a special case ($\beta = \alpha$ and $N = M$) of the generalized Jacobi polynomials $\left\{P_n^{\alpha, \beta, M, N}(x)\right\}_{n=0}^{\infty}$, but the differential equation (18) has a very special form without a M^2 -part. This is explained by the fact that

$$\begin{cases} C_0 = \left[1 + 2M \binom{n + 2\alpha + 1}{n - 1}\right]^2 \\ C_1 = \frac{2M}{2\alpha + 1} \binom{n + 2\alpha}{n} \left[1 + 2M \binom{n + 2\alpha + 1}{n - 1}\right] \end{cases}.$$

This implies that the generalized ultraspherical polynomials satisfy the same differential equation as the polynomials $\left\{Q_n^{\alpha, \alpha, M, M}(x)\right\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Q_n^{\alpha, \alpha, M, M}(x) = & \left[1 + 2M \binom{n + 2\alpha + 1}{n - 1}\right] P_n^{(\alpha, \alpha)}(x) + \\ & - \frac{2M}{2\alpha + 1} \binom{n + 2\alpha}{n} x D P_n^{(\alpha, \alpha)}(x), \quad n = 0, 1, 2, \dots \end{aligned}$$

However, this differential equation will appear not to be a special case of the differential equation of the form (2) for the generalized Jacobi polynomials, since the MN -part will not vanish if we take $\beta = \alpha$ and $N = M$. We aim to give a proof of this in a future publication. In [7] we applied the special case $\beta = \alpha$ of the Jacobi inversion formula (17) to solve the systems of equations obtained in [11].

4 The inversion formulas

In [4] H. Bavinck and R. Koekoek found the following inversion formula involving Charlier polynomials

$$\sum_{k=j}^i C_{i-k}^{(-a)}(-x) C_{k-j}^{(a)}(x) = \delta_{ij}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots \quad (19)$$

This formula is an easy consequence of the generating function (see for instance [13])

$$e^{-at}(1+t)^x = \sum_{n=0}^{\infty} C_n^{(a)}(x) t^n.$$

In fact we have

$$\begin{aligned} 1 &= e^{-at}(1+t)^x e^{at}(1+t)^{-x} = \sum_{k=0}^{\infty} C_k^{(a)}(x) t^k \sum_{m=0}^{\infty} C_m^{(-a)}(-x) t^m \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n C_k^{(a)}(x) C_{n-k}^{(-a)}(-x) \right) t^n. \end{aligned}$$

Hence

$$\sum_{k=0}^n C_k^{(a)}(x) C_{n-k}^{(-a)}(-x) = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, 3, \dots \end{cases}$$

Now (19) easily follows by taking $n = i - j$ and shifting the summation index. This formula was also used in [2] to find difference operators with Sobolev-type Charlier polynomials as eigenfunctions. In [3] a similar formula involving Meixner polynomials was used to find difference equations for generalized Meixner polynomials.

Formula (19) can be interpreted as follows. If we define the matrix $T = (t_{ij})_{i,j=0}^n$ with entries

$$t_{ij} = \begin{cases} C_{i-j}^{(a)}(x), & j \leq i \\ 0, & j > i, \end{cases}$$

then this matrix T is a triangular matrix with determinant 1 and the inverse U of this matrix is given by $T^{-1} = U = (u_{ij})_{i,j=0}^n$ with entries

$$u_{ij} = \begin{cases} C_{i-j}^{(-a)}(-x), & j \leq i \\ 0, & j > i. \end{cases}$$

Therefore we call (19) an inversion formula.

In the same way we find by using the generating function (4) for the Laguerre polynomials

$$\sum_{k=j}^i L_{i-k}^{(\alpha)}(x) L_{k-j}^{(-\alpha-2)}(-x) = \delta_{ij}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots \quad (20)$$

However, this formula cannot be used to solve systems of equations of the form

$$\sum_{i=1}^{\infty} A_i(x) D^i L_n^{(\alpha)}(x) = F_n(x), \quad n = 1, 2, 3, \dots$$

in view of the parametershift in (5).

In [1] H. Bavinck used a slightly different method to find the Laguerre inversion formula (16) from the generating function (4) for the Laguerre polynomials. In fact we have

$$\begin{aligned}
(1-t)^{i-j-1} &= (1-t)^{-\alpha-j-1} \exp\left(\frac{xt}{t-1}\right) (1-t)^{\alpha+i} \exp\left(\frac{-xt}{t-1}\right) \\
&= \sum_{k=0}^{\infty} L_k^{(\alpha+j)}(x) t^k \sum_{m=0}^{\infty} L_m^{(-\alpha-i-1)}(-x) t^m \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n L_k^{(\alpha+j)}(x) L_{n-k}^{(-\alpha-i-1)}(-x) \right) t^n.
\end{aligned}$$

This implies, by comparing the coefficients of t^{i-j} on both sides, that

$$\sum_{k=0}^{i-j} L_k^{(\alpha+j)}(x) L_{i-j-k}^{(-\alpha-i-1)}(-x) = \delta_{ij}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots,$$

which is equivalent to (16).

Formula (16) can be interpreted as follows. If we define the matrix $T = (t_{ij})_{i,j=0}^n$ with entries

$$t_{ij} = \begin{cases} L_{i-j}^{(\alpha+j)}(x), & j \leq i \\ 0, & j > i, \end{cases}$$

then this matrix T is a triangular matrix with determinant 1 and the inverse U of this matrix is given by $T^{-1} = U = (u_{ij})_{i,j=0}^n$ with entries

$$u_{ij} = \begin{cases} L_{i-j}^{(-\alpha-i-1)}(-x), & j \leq i \\ 0, & j > i. \end{cases}$$

In case of the Jacobi polynomials the above methods seem not to be applicable. In that case we have to find the inverse of the matrix $T = (t_{ij})_{i,j=0}^n$ with entries

$$t_{ij} = D^j P_i^{(\alpha,\beta)}(x), \quad i, j = 0, 1, 2, \dots, n.$$

This matrix T is also triangular and by using (11) the diagonal entries equal

$$t_{ii} = D^i P_i^{(\alpha,\beta)}(x) = \frac{(i + \alpha + \beta + 1)_i}{2^i}, \quad i = 0, 1, 2, \dots, n.$$

This implies that the determinant of T is nonzero for each n iff $-(\alpha + \beta + 2) \notin \{0, 1, 2, \dots\}$. In that case T is invertible and if the inverse U is given by $T^{-1} = U = (u_{ij})_{i,j=0}^n$ then we must have

$$u_{ii} = \frac{1}{t_{ii}} = \frac{2^i}{(\alpha + \beta + i + 1)_i}, \quad i = 0, 1, 2, \dots, n.$$

In the next section we will give a proof of the Jacobi inversion formula (17), which is equivalent to

$$u_{ij} = \begin{cases} \frac{(\alpha + \beta + 2j + 1)2^i}{(\alpha + \beta + j + 1)_{i+1}} P_{i-j}^{(-\alpha-i-1, -\beta-i-1)}(x), & j \leq i \\ 0, & j > i. \end{cases}$$

5 Proof of the Jacobi inversion formula

In this section we will prove that

$$\begin{aligned} & \sum_{k=0}^n \frac{(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_k}{\Gamma(\alpha + \beta + n + k + 2)} P_k^{(\alpha, \beta)}(x) P_{n-k}^{(-n-\alpha-1, -n-\beta-1)}(y) \\ &= \frac{1}{\Gamma(\alpha + \beta + 1)} \frac{1}{n!} \left(\frac{x-y}{2} \right)^n, \quad n = 0, 1, 2, \dots, \end{aligned} \quad (21)$$

which holds for all α and β .

Note that (14) is a special case of (21) since

$$\begin{aligned} P_{n-k}^{(-n-\alpha-1, -n-\beta-1)}(1) &= \frac{(-n-\alpha)_{n-k}}{(n-k)!} = (-1)^{n-k} \frac{(\alpha + k + 1)_{n-k}}{(n-k)!} \\ &= \frac{(-1)^n}{n!} (-n)_k (\alpha + k + 1)_{n-k}, \quad k = 0, 1, 2, \dots, n \end{aligned}$$

for all $n \in \{0, 1, 2, \dots\}$.

By taking $y = x$ in (21) we easily obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_k}{\Gamma(\alpha + \beta + n + k + 2)} \times \\ & \times P_k^{(\alpha, \beta)}(x) P_{n-k}^{(-n-\alpha-1, -n-\beta-1)}(x) = \begin{cases} \frac{1}{\Gamma(\alpha + \beta + 1)}, & n = 0 \\ 0, & n = 1, 2, 3, \dots \end{cases} \end{aligned} \quad (22)$$

for all α and β . If we take $n = i - j$ in (22) and shift the summation index we find

$$\begin{aligned} & \sum_{k=j}^i \frac{(\alpha + \beta + 2k - 2j + 1)(\alpha + \beta + 1)_{k-j}}{\Gamma(\alpha + \beta + i - 2j + k + 2)} \times \\ & \times P_{i-k}^{(-i+j-\alpha-1, -i+j-\beta-1)}(x) P_{k-j}^{(\alpha, \beta)}(x) = \frac{\delta_{ij}}{\Gamma(\alpha + \beta + 1)}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots \end{aligned}$$

For α and β real with $\alpha + \beta + 1 > -1$ we now obtain (17) by shifting both α and β by j .

Note that (21) for $y = -x$ in a similar way leads to

$$\begin{aligned} & \sum_{k=j}^i \frac{\alpha + \beta + 2k + 1}{(\alpha + \beta + k + j + 1)_{i-j+1}} \times \\ & \times P_{i-k}^{(-\alpha-i-1, -\beta-i-1)}(-x) P_{k-j}^{(\alpha+j, \beta+j)}(x) = \frac{x^{i-j}}{(i-j)!}, \quad j \leq i, \quad i, j = 0, 1, 2, \dots \end{aligned}$$

This formula was used in [7].

In order to prove (21) we start with the left-hand side, apply definition (8) to $P_k^{(\alpha, \beta)}(x)$ and definition (9) to $P_{n-k}^{(-n-\alpha-1, -n-\beta-1)}(y)$ and change the order of summation to obtain

$$\sum_{k=0}^n \frac{(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_k}{\Gamma(\alpha + \beta + n + k + 2)} P_k^{(\alpha, \beta)}(x) P_{n-k}^{(-n-\alpha-1, -n-\beta-1)}(y)$$

$$\begin{aligned}
&= \sum_{k=0}^n \sum_{i=0}^k \sum_{j=0}^{n-k} (-1)^{n-k} \frac{(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_k}{\Gamma(\alpha + \beta + n + k + 2)} \frac{(\alpha + \beta + k + 1)_i}{i!} \frac{(\alpha + i + 1)_{k-i}}{(k-i)!} \times \\
&\quad \times \frac{(\alpha + \beta + n + k - j + 2)_j}{j!} \frac{(\alpha + k + 1)_{n-k-j}}{(n-k-j)!} \left(\frac{x-1}{2}\right)^i \left(\frac{y-1}{2}\right)^j \\
&= \sum_{i=0}^n \sum_{k=i}^n \sum_{j=0}^{n-k} (-1)^{n-k} \times \\
&\quad \times \frac{(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_{i+k}(\alpha + i + 1)_{n-i-j}}{\Gamma(\alpha + \beta + n + k - j + 2) i! (k-i)! j! (n-k-j)!} \left(\frac{x-1}{2}\right)^i \left(\frac{y-1}{2}\right)^j \\
&= \sum_{i=0}^n \sum_{k=0}^{n-i} \sum_{j=0}^{n-i-k} (-1)^{n-i-k} \times \\
&\quad \times \frac{(\alpha + \beta + 2i + 2k + 1)(\alpha + \beta + 1)_{2i+k}(\alpha + i + 1)_{n-i-j}}{\Gamma(\alpha + \beta + n + i + k - j + 2) i! k! j! (n-i-k-j)!} \left(\frac{x-1}{2}\right)^i \left(\frac{y-1}{2}\right)^j \\
&= \sum_{i=0}^n \sum_{j=0}^{n-i} \sum_{k=0}^{n-i-j} (-1)^{n-i} \left(\frac{x-1}{2}\right)^i \left(\frac{y-1}{2}\right)^j \times \\
&\quad \times \frac{(\alpha + \beta + 2i + 2k + 1)(\alpha + \beta + 1)_{2i+k}(\alpha + i + 1)_{n-i-j}(-n+i+j)_k}{\Gamma(\alpha + \beta + n + i - j + k + 2) i! j! k! (n-i-j)!} \\
&= \sum_{i=0}^n \sum_{j=0}^{n-i} (-1)^{n-i} \frac{(\alpha + \beta + 1)_{2i}(\alpha + i + 1)_{n-i-j}}{i! j! (n-i-j)!} \left(\frac{x-1}{2}\right)^i \left(\frac{y-1}{2}\right)^j \times \\
&\quad \times \sum_{k=0}^{n-i-j} \frac{(-n+i+j)_k(\alpha + \beta + 2i + 1)_k}{\Gamma(\alpha + \beta + n + i - j + k + 2) k!} (\alpha + \beta + 2i + 2k + 1), \quad n = 0, 1, 2, \dots
\end{aligned}$$

Now we will show that for all b we have

$$\sum_{k=0}^n \frac{(-n)_k (b)_k}{\Gamma(b + n + k + 1) k!} (b + 2k) = 0, \quad n = 1, 2, 3, \dots \quad (23)$$

In order to prove this we use the well-known Vandermonde summation formula

$${}_2F_1 \left(\begin{matrix} -n, b \\ c \end{matrix} \middle| 1 \right) = \frac{(c-b)_n}{(c)_n}, \quad (c)_n \neq 0, \quad n = 0, 1, 2, \dots,$$

which can be written in a more general form as

$$\sum_{k=0}^n \frac{(-n)_k (b)_k}{\Gamma(c + k) k!} = \frac{(c-b)_n}{\Gamma(c + n)}, \quad n = 0, 1, 2, \dots$$

This formula is valid for all b and c . By using this we find that for all b we have

$$\begin{aligned}
&\sum_{k=0}^n \frac{(-n)_k (b)_k}{\Gamma(b + n + k + 1) k!} (b + 2k) \\
&= b \sum_{k=0}^n \frac{(-n)_k (b + 1)_k}{\Gamma(b + n + k + 1) k!} - nb \sum_{k=0}^{n-1} \frac{(-n + 1)_k (b + 1)_k}{\Gamma(b + n + k + 2) k!} \\
&= b \frac{(n)_n}{\Gamma(b + 2n + 1)} - nb \frac{(n + 1)_{n-1}}{\Gamma(b + 2n + 1)} = 0, \quad n = 1, 2, 3, \dots,
\end{aligned}$$

which proves (23). Now we use (23) to obtain

$$\begin{aligned}
& \sum_{k=0}^n \frac{(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_k}{\Gamma(\alpha + \beta + n + k + 2)} P_k^{(\alpha, \beta)}(x) P_{n-k}^{(-n-\alpha-1, -n-\beta-1)}(y) \\
&= \sum_{i=0}^n (-1)^{n-i} \frac{(\alpha + \beta + 1)_{2i}}{i!(n-i)!} \left(\frac{x-1}{2}\right)^i \left(\frac{y-1}{2}\right)^{n-i} \frac{\alpha + \beta + 2i + 1}{\Gamma(\alpha + \beta + 2i + 2)} \\
&= \frac{1}{\Gamma(\alpha + \beta + 1)} \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} \left(\frac{x-1}{2}\right)^i \left(\frac{1-y}{2}\right)^{n-i} \\
&= \frac{1}{\Gamma(\alpha + \beta + 1)} \frac{1}{n!} \left(\frac{x-y}{2}\right)^n, \quad n = 0, 1, 2, \dots,
\end{aligned}$$

which proves (21).

6 Some remarks

Note that we have from definition (3) for the Laguerre polynomials that

$$L_n^{(-n)}(x) = (-1)^n \frac{x^n}{n!}, \quad n = 0, 1, 2, \dots \quad (24)$$

Hence, the polynomial $L_n^{(-n)}(x)$ reduces to a monomial of degree n for all $n \in \{0, 1, 2, \dots\}$. Definition (10) for the Jacobi polynomials leads to

$$P_n^{(-n, \beta)}(x) = \binom{n + \beta}{n} \left(\frac{x-1}{2}\right)^n, \quad n = 0, 1, 2, \dots,$$

which is also a monomial. However, this monomial might reduce to the zero polynomial. For instance, $P_n^{(-n, -n)}(x)$ equals the zero polynomial for all $n \in \{1, 2, 3, \dots\}$.

It is possible to generalize the Laguerre inversion formula (16) to

$$\sum_{k=0}^n L_k^{(\alpha+p_n)}(x) L_{n-k}^{(-\alpha-q_n)}(-x) = \frac{(p_n - q_n + 2)_n}{n!}, \quad n = 0, 1, 2, \dots, \quad (25)$$

where p_n and q_n are arbitrary and even may depend on n . In order to have an inversion formula we have to choose p_n and q_n such that

$$(p_n - q_n + 2)_n = 0, \quad n = 1, 2, 3, \dots,$$

hence

$$p_n - q_n \in \{-n-1, -n, \dots, -3, -2\}, \quad n = 1, 2, 3, \dots$$

Note that the endpoint-cases $p_n - q_n = -n-1$ and $p_n - q_n = -2$ correspond to the earlier mentioned inversion formulas (16) and (20) respectively.

To prove (25) we use (4) to obtain

$$L_n^{(\alpha)}(x) = \frac{1}{n!} D_t^n \left[(1-t)^{-\alpha-1} \exp\left(\frac{xt}{t-1}\right) \right] \Big|_{t=0}, \quad n = 0, 1, 2, \dots,$$

where $D_t = \frac{d}{dt}$ denotes differentiation with respect to t . Hence by using Leibniz' rule we find

$$\begin{aligned}
& \sum_{k=0}^n L_k^{(\alpha+p_n)}(x) L_{n-k}^{(-\alpha-q_n)}(-x) \\
&= \sum_{k=0}^n \frac{1}{k!} D_t^k \left[(1-t)^{-\alpha-p_n-1} \exp\left(\frac{xt}{t-1}\right) \right] \Big|_{t=0} \times \\
&\quad \times \frac{1}{(n-k)!} D_t^{n-k} \left[(1-t)^{\alpha+q_n-1} \exp\left(\frac{-xt}{t-1}\right) \right] \Big|_{t=0} \\
&= \frac{1}{n!} D_t^n \left[(1-t)^{q_n-p_n-2} \right] \Big|_{t=0} = \frac{(p_n - q_n + 2)_n}{n!}, \quad n = 0, 1, 2, \dots,
\end{aligned}$$

which proves (25).

Further we remark that if we replace x by $1 - \frac{2x}{\beta}$ and y by $1 - \frac{2y}{\beta}$ in (21), multiply by $\Gamma(\alpha + \beta + 1)\beta^n$ and let β tend to infinity in an appropriate way we obtain by using (15)

$$\sum_{k=0}^n L_k^{(\alpha)}(x) L_{n-k}^{(-n-\alpha-1)}(-y) = \frac{(y-x)^n}{n!}, \quad n = 0, 1, 2, \dots \quad (26)$$

Note that (7) is a special case of (26) since

$$L_{n-k}^{(-n-\alpha-1)}(0) = (-1)^{n-k} \frac{(\alpha + k + 1)_{n-k}}{(n-k)!} = (-1)^n (-1)^k \binom{n+\alpha}{n-k}, \quad k = 0, 1, 2, \dots, n$$

for all $n \in \{0, 1, 2, \dots\}$. Moreover, note that (26) is a special case of the well-known convolution formula for the classical Laguerre polynomials

$$\sum_{k=0}^n L_k^{(\alpha)}(x) L_{n-k}^{(\beta)}(y) = L_n^{(\alpha+\beta+1)}(x+y), \quad n = 0, 1, 2, \dots$$

in view of (24). By using the technique demonstrated above this convolution formula can be proved for all α and β which might even depend on n .

Finally we remark that, by using the fact that

$$(b/2)_k (b+2k) = b(b/2+1)_k, \quad k = 0, 1, 2, \dots,$$

formula (23) can also be obtained by using a summation formula for a terminating well-poised hypergeometric series (see for instance formula (III.9) in [16]).

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